TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 340, Number 1, November 1993

SINGULAR INTEGRAL OPERATORS ON C^1 MANIFOLDS

JEFF E. LEWIS, RENATA SELVAGGI, AND IRENE SISTO

ABSTRACT. We show that the kernel of a singular integral operator is real analytic in $\mathbf{R}^n \setminus \{0\}$ iff the symbol [Fourier transform] is real analytic in $\mathbf{R}^n \setminus \{0\}$. The singular integral operators with continuous coefficients and real analytic kernels (symbols) form an operator algebra with the usual symbolic calculus. The symbol is invariantly defined under C^1 changes of coordinates.

Algebras of singular integral operators (s.i.o.) or pseudodifferential operators have been studied from various points of view. For the C^∞ theory, the pseudodifferential operator approach using Fourier transforms or oscillating integrals gives a full asymptotic expansion of the operator, but does not give minimal assumptions on the coefficients for partial results. Let T be a singular integral operator, e.g., the Hilbert transform, and $a(x)$ a bounded function with compact support. Consider the commutator

$$C = aT - Ta.$$

Then C is a compact operator on L^p under the sole hypothesis that $a(x)$ is continuous. The continuity assumption cannot be relaxed; in dimension $n = 1$, a jump discontinuity in $a(x)$ makes C a Hardy kernel operator which is not compact on $L^p(\mathbf{R})$ and the symbolic calculus must include Hardy kernels [Cos; LP; L].

In this note we develop an algebra, $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$, of s.i.o. which have continuous coefficients and are invariantly defined under C^1 changes of coordinates. The price to pay is to assume real analyticity of the kernels [symbols] in the convolution [Fourier multiplier] variables. The first observation is that a s.i.o. has a real analytic kernel iff its symbol [Fourier transform] is real analytic. For the C^1 change of coordinates, we take the approach of s.i.o. and apply the method of rotation of Calderón and Zygmund [CZ 1] for odd kernels. We use the full power of the Multilinear Commutator Theorem of Coifman, McIntosh, and Meyer [CMM, Theorem 3]. The algebras of s.i.o. may be transported to a C^1 manifold $\mathscr{M}$, as in Seeley [SCM] for manifolds of class C^k, $k \geq 3$.

We shall follow the notation of [CZ 1] and [SCM] with a few exceptions. Points in $\mathbf{R}^n$ will be denoted $x = (x_1, \dots, x_n)$, etc.; $x' = x/|x|$. The unit sphere $\{|x| = 1\}$ will be denoted by Σ and its element of surface area by $d\sigma$.

Received by the editors March 21, 1991 and, in revised form, September 6, 1991.
1991 *Mathematics Subject Classification*. Primary 35S15, 45E05.
Key words and phrases. Layer potentials, nonsmooth domains, pseudodifferential operators, singular integrals, symbolic calculus.
Research partially supported by the Italian CNR.

The space $C^\beta(\mathbf{R}^n)$, $\beta \geq 0$, will denote the class of complex functions on $\mathbf{R}^n$ whose derivatives up to order $[\beta]$ are bounded and continuous on $\mathbf{R}^n$ and whose derivatives of order $[\beta]$ satisfy a uniform Hölder condition of order $\beta - [\beta]$. For $1 < p < \infty$, L^p will denote the space of measurable functions whose pth power is integrable, and for k a nonnegative integer, L^p_k is the space of functions whose derivatives of order $\leq k$ are in L^p; the norm on L^p_k is

$$\|f\|_{p,k} = \Big(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_p^p \Big)^{1/p}$$

The Fourier transform is defined as $\hat{f}(\xi) = \int e^{-ix\xi} f(x)\, dx$, so that $f(x) = (2\pi)^{-n} \int e^{in\xi} \hat{f}(\xi)\, d\xi$.

A homogeneous function $F(z)$ is in $\mathscr{O}$ if it is real analytic on $\mathbf{R}^n \setminus \{0\}$; if $F(z) \in \mathscr{O}$, then there are constants C, $R > 0$ such that for $1 \leq |z| \leq 2$,

$$|D^\alpha F(z)| \leq C(1/R)^{|\alpha|} \alpha!.$$

From [SCM] we recall the definitions of functions homogeneous of degree 0 and $-n$.

Definition 0.1. Let $F(x, z)$ be defined on $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$. We say that $F(x, z)$ *is homogeneous of degree* 0 if

$$F(x, \lambda z) = F(x, z) = F(x, z'), \quad \text{for all } \lambda > 0,\ z \neq 0.$$

We say that $F(x, z)$ *is homogeneous of degree* $-n$ if

$$F(x, \lambda z) = \lambda^{-n} F(x, z) = |\lambda z|^{-n} F(x, z'), \quad \text{for all } \lambda > 0,\ z \neq 0,$$

and

$$\int_\Sigma F(x, z')\, d\sigma = 0.$$

The following definition describes the kernel and symbol classes for our s.i.o.

Definition 0.2. Let $F(x, z)$ be homogeneous of degree 0 or of degree $-n$. We say that $F(x, z) \in \mathscr{C}^{\beta,\infty}(\mathbf{R}^n)$ iff $F(x, z)$ is C^β with respect to x, uniformly for $1 \leq |z| \leq 2$, and C^∞ with respect to z, uniformly in x.

We say that $F(x, z) \in \mathscr{C}^{\beta,\mathscr{O}}(\mathbf{R}^n)$ iff $F(x, z)$ is C^β with respect to x, uniformly for $1 \leq |z| \leq 2$, and real analytic with respect to z, $1 \leq |z| \leq 2$, uniformly in x.

In the case that $F(x, z) = F(z)$ is independent of x, we shall sometimes write $F(z) \in \mathscr{C}^{\cdot,\infty}(\mathbf{R}^n)$ or $F(z) \in \mathscr{C}^{\cdot,\mathscr{O}}(\mathbf{R}^n)$.

In §1 we study homogeneous functions of degree 0 which are real analytic. Let $F(z) = F(z')$ be homogeneous of degree 0 and let

$$F(z') = a_0 + \sum a_{lm} Y_{lm}(z') = a_0 + \sum_{m=1}^{\infty} \sum_{l=1}^{d(m)} a_{lm} Y_{lm}(z')$$

be its expansion in spherical harmonics. We show that $F(z)$ is real analytic in $\mathbf{R}^n \setminus \{0\}$ iff the sequence $\{a_{lm}\}$ is exponentially decreasing, i.e., there are constants c, θ, $0 \leq \theta < 1$, such that

$$|a_{lm}| \leq c\, \theta^m.$$

As a consequence (Theorem 2) a s.i.o. has a real analytic kernel iff its symbol is also real analytic and the composition of two convolution s.i.o. with real analytic kernels has a real analytic kernel.

In §2 we treat algebras of s.i.o. on $\mathbf{R}^n$. The symbols are homogeneous of degree 0 and in $\mathscr{C}^{k,\mathscr{O}}(\mathbf{R}^n)$; the kernels are homogeneous of degree $-n$ and in $\mathscr{C}^{k,\mathscr{O}}(\mathbf{R}^n)$. The operators to be ignored are the compact operators on L^p (more precisely see Definition 2.1). The Commutator Theorems and the Symbol Exact Sequence (Theorem 3) are stated without proofs since our operators in $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ form a subalgebra of the s.i.o. of type C_0^∞ in the notation of [SCM].

In §3 we treat C^1 coordinate changes for operators in $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$. The crucial result (Theorem 5) is an approximation theorem for odd kernel operators. The real analyticity of the kernels is required to apply the Multilinear Commutator Theorem of [CMM].

With the result of Theorem 4, in §4 we indicate how the program of Seeley [SCM] may be carried out to transplant $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ to a compact C^1 manifold.

An example. Consider a double layer potential on the boundary of a C^1 domain in $\mathbf{R}^{n+1}$. Let Ω be a C^1 domain in $\mathbf{R}^{n+1}$ and $k(X)$ an analytic double layer kernel:

(1) $k(\lambda X) = \lambda^{-n} k(X)$, $\lambda > 0$, $X \in \mathbf{R}^{n+1} \setminus \{0\}$,
(2) k is odd: $k(-X) = -k(X)$, $X \neq 0$,
(3) k is real analytic in $\mathbf{R}^{n+1} \setminus \{0\}$.

If $a(P)$ and $b(P)$ are continuous on $\partial\Omega$, then

$$Tf(P) = a(P)f(P) + \mathrm{p.v.} \int_{\partial\Omega} k(P-Q)f(Q)b(Q)\, d\sigma_Q$$

is an operator in $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\partial\Omega)$. These layer potentials on C^1 surfaces in $\mathbf{R}^{n+1}$, $n \geq 2$, are considered by Selvaggi and Sisto in [SS 1] and [SS 2]. Bounds for the nontangential maximal function are shown in [SS 1]. In [SS 2], it is shown that the operator T differs locally from an s.i.o. by a compact operator.

The pioneering work on potentials on C^1 domains was done by Fabes, Jodeit, and Rivière [FJR]. Cohen and Gosselin [CG] treated multiple layer potentials for the biharmonic operator on C^1 domains. In [FJR] and [CG] the layer potentials are of the form $I + C$, C compact, so that the symbols are the identity.

For a pseudodifferential operator approach to operators with analytic coefficients and analytic kernels see Boutet de Monvel and Krée [BK, B 1, B 2].

The authors thank A. Bove for several discussions regarding this work.

1. Real analytic homogeneous functions and spherical harmonics

We recall some facts about spherical harmonics [CZ 1; N, pp. 212–237]. Let $Y_m(z)$ be a normalized real spherical harmonic of degree m: $Y_m(z) = Y_m(z') = |z|^{-m} P_m(z)$ where $P_m(z)$ is a homogeneous polynomial of degree m satisfying $\Delta P_m = 0$ in $\mathbf{R}^n$ and $\int_\Sigma |Y_m(z')|^2\, d\sigma = 1$. We let $d(m)$ denote the dimension of the space of spherical harmonics of degree m and let $Y_{lm}(z')$,

$l = 1, \dots, d(m)$, be an orthonormal basis for the spherical harmonics of degree m. We have that $d(m) \sim 2m^{n-2}/(n-2)!$ [N, p. 218]. If $F(z) = F(z')$ is a function homogeneous of degree 0 and $F(z') \in L^2(\Sigma)$, we let

$$F(z') \sim a_0 + \sum a_{lm} Y_{lm}(z') = a_0 + \sum_{m=1}^{\infty} \sum_{l=1}^{d(m)} a_{lm} Y_{lm}(z') \tag{1–1}$$

be its expansion in spherical harmonics; here

$$a_0 = \frac{1}{|\Sigma|} \int_\Sigma F(z')\, d\sigma, \qquad a_{lm} = \int_\Sigma F(z') Y_{lm}(z')\, d\sigma. \tag{1–2}$$

It is known that $F(z') \in \mathscr{E}^{\cdot,\infty}(\mathbf{R}^n)$ iff the sequence of coefficients in (1–1) is rapidly decreasing; i.e., for every r, $|a_{lm}| \le c_r m^{-r}$. We now characterize functions in $\mathscr{E}^{\cdot,\mathscr{O}}(\mathbf{R}^n)$ in terms of exponentially decreasing coefficients.

Theorem 1. *Let* $F(z') \in \mathscr{E}^{\cdot,\infty}(\mathbf{R}^n)$ *and let*

$$F(z') = a_0 + \sum a_{lm} Y_{lm}(z') \tag{1–3}$$

be its expansion in spherical harmonics. Then $F(z') \in \mathscr{E}^{\cdot,\mathscr{O}}(\mathbf{R}^n)$ *iff there is a constant* c *and a number* θ, $0 \le \theta < 1$, *such that*

$$|a_{lm}| \le c\,\theta^m. \tag{1–4}$$

Proof. Let $F(z')$ be real analytic. If $G(z')$ is homogeneous of degree 0, let $LG(z) = |z|^2 \Delta G(z) = \Delta_\Sigma G(z')$, where Δ_Σ is the Laplace-Beltrami operator on Σ. Then [CZ 1, p. 905] for every integer $r > 0$,

$$a_{lm} = \frac{(-1)^r}{m^r(m+n-2)^r} \int_\Sigma L^r F(z')\, Y_{lm}(z')\, d\sigma. \tag{1–5}$$

Since $|\Delta_\Sigma^r F(z')| \le c(1/R)^{2r}(2r)!$ and $|Y_{lm}(z')| \le c\, m^{n/2-1}$,

$$|a_{lm}| \le c\, m^{n/2-1-2r}(1/R)^{2r}!. \tag{1–6}$$

By Stirling's formula $(2r)! \sim \sqrt{4\pi r}(2r/e)^{2r}$; for m large choose $2r \sim Rm$. Then

$$|a_{lm}| \le c\, m^{n/2-1/2}(Rm/Rme)^{2r} \sim c\, m^{n/2-1/2}\left[e^{-R}\right]^m \le c_\theta \theta^m$$

if $e^{-R} < \theta < 1$.

Suppose that (1–4) holds and choose $\delta > 0$ so small that $\theta(1-\delta)^{-1} = \theta_1 < 1$. We first estimate $|D^\alpha Y_m(z)|$. Let $Y_m(z) = |z|^{-m} P_m(z)$. By the estimate [CZ 1, p. 903]

$$|D^\alpha P_m(z)| \le C c^{|\alpha|} m^{n/2-1+|\alpha|} |z|^{m-|\alpha|}. \tag{1–7}$$

Let $g_m(z) = |z|^{-m}$. Since g_m is real analytic in $\mathbf{R}^n \setminus \{0\}$, near $|z| = 1$, we use the contour integral representation of $g_m(z)$ along the contours $|\zeta_i - z_i| = \delta$ to obtain that

$$|D^\alpha(|z|^{-m})| \le c(1-\delta)^{-m} \delta^{-n-|\alpha|} \alpha!. \tag{1–8}$$

Using Leibniz,

$$D^\alpha Y_m(z) = D^\alpha(|z|^{-m} P_m(z)) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} D^\gamma(|z|^{-m}) D^{\alpha-\gamma} P_m(z). \tag{1–9}$$

Thus

$$(1\text{–}10)\quad |D^\alpha Y_m(z)| \le c\,\delta^{-n} m^{n/2-1} |z|^{-|\alpha|} (1-\delta)^{-m} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \gamma!\,\delta^{-|\gamma|} c^{|\alpha-\gamma|} m^{|\alpha-\gamma|}.$$

The trick is to observe that for fixed γ, $\gamma \le \alpha$,

$$(1\text{–}11)\qquad \gamma!\, m^{|\alpha-\gamma|} \le \sum_{\gamma\le\alpha} \gamma!\, m^{|\alpha-\gamma|} \le m(m+1)\cdots(m+|\alpha|).$$

In fact if $|\alpha| = N$, the sum in (1–11) is maximized when $|\alpha| = (N, 0, \dots, 0)$ and in this case

$$(1\text{–}12)\qquad \sum_{j=0}^{N} j!\, m^{N-j} = m^{N+1} + \sum_{j=1}^{N} j!\, m^{N-j} \le m^{N+1} + (m+1)\cdots(m+N).$$

The inequality in (1–12) is easily seen by comparing the coefficients of the polynomials in m. It follows that

$$(1\text{–}13)\qquad \begin{aligned} |D^\alpha Y_m(z)| \le c\,\delta^{-n} m^{\frac{n}{2}} |z|^{-|\alpha|} (1-\delta)^{-m} (\delta^{-1}+c)^{|\alpha|} \\ \times (m+1)(m+2)\cdots(m+|\alpha|). \end{aligned}$$

Using (1–4) and (1–13),
(1–14)

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{l=1}^{d(m)} |a_{lm}| |D^\alpha Y_{lm}(z')| \\ &\qquad \le c_\delta (\delta^{-1}+c)^{|\alpha|} \sum_{m=1}^{\infty} m^{n/2+n-2} (\theta(1-\delta)^{-1})^m (m+1)(m+2)\cdots(m+|\alpha|), \end{aligned}$$

which is dominated by

$$(1\text{–}15)\qquad c_\delta (\delta^{-1}+c)^{|\alpha|} \sum_{m=1}^{\infty} {\theta_2}^m (m+1)\cdots(m+|\alpha|),$$

where $\theta(1-\delta)^{-1} = \theta_1 < \theta_2 < 1$. The sum on the right-hand side of (1–14) is dominated by $|\alpha|!\,(1-\theta_2)^{-|\alpha|-1}$. Thus we have

$$(1\text{–}16)\qquad |D^\alpha F(z')| \le c_\delta (1/R)^{|\alpha|} |\alpha|!$$

with $R^{-1} = (\delta^{-1}+c)(1-\theta_2)^{-1}$, so that $F(z')$ is real analytic in $\mathbf{R}^n \setminus \{0\}$. □

Following [CZ 1], we define a s.i.o. via spherical harmonics. If $Y_{lm}(z')$ is a spherical harmonic, define the s.i.o. kernel k_{lm} and operator T_{lm} on $L^p(\mathbf{R}^n)$ by

$$(1\text{–}17)\qquad \begin{aligned} k_{lm}(z) &= |z|^{-n} Y_{lm}(z'), \\ T_{lm} f(x) &= \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} k_{lm}(x-y) f(y)\, dy \\ &= \frac{1}{(2\pi)^n} \int e^{ix\xi} \sigma(T_{lm})(\xi) \hat{f}(\xi)\, d\xi, \end{aligned}$$

where $\hat{f}(\xi)$ is the Fourier transform of f and $\sigma(T_{lm})(\xi)$ is Fourier transform of the distribution kernel p.v. k_{lm}:

$$\sigma(T_{lm})(\xi) = \lim_{\epsilon \to 0,\, N \to \infty} \int_{\epsilon < |x| < N} e^{-iz\xi} k_{lm}(z)\, dz = \gamma_m Y_{lm}(\xi'), \tag{1-18}$$
$$\gamma_m = \frac{(-i)^m \pi^{n/2} \Gamma(m/2)}{\Gamma((m+n)/2)}.$$

Now let $k(x, z) \in \mathscr{C}^{\beta, \infty}(\mathbf{R}^n)$ be homogeneous of degree $-n$,

$$k(x, z') = \sum a_{lm}(x) Y_{lm}(z');$$

define the s.i.o. T on $L^p(\mathbf{R}^n)$ by

$$\begin{aligned} Tf(x) &= \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x, x-y) f(y)\, dy \\ &= \sum a_{lm}(x) T_{lm} f(x) \\ &= \frac{1}{(2\pi)^n} \int e^{ix\xi} \sigma(T)(x, \xi) \hat{f}(\xi)\, d\xi, \end{aligned} \tag{1-19}$$

where

$$\begin{aligned} \sigma(T)(x, \xi) &= \lim_{\epsilon \to 0,\, N \to \infty} \int_{\epsilon < |x| < N} e^{-iz\xi} k(x, z)\, dz \\ &= \sum a_{lm}(x) \gamma_m Y_{lm}(\xi'). \end{aligned} \tag{1-20}$$

From Theorem 1, we have

Theorem 2. *Let the operator T of* (1–19) *have a kernel $k \in \mathscr{C}^{\beta, \infty}(\mathbf{R}^n)$. Then the symbol of T, $\sigma(T)(x, \xi)$, defined by* (1–20), *which is homogeneous of degree* 0 *and in $\mathscr{C}^{\beta, \infty}(\mathbf{R}^n)$, is also in $\mathscr{C}^{\beta, \mathscr{O}}(\mathbf{R}^n)$ iff the kernel k is in $\mathscr{C}^{\beta, \mathscr{O}}(\mathbf{R}^n)$.*

Remark. Theorem 2 could be proved by using the result of Boutet de Monvel and Krée [BK, Proposition 0.4] that the Fourier transform of a holomorphic distribution homogeneous of degree α is a holomorphic distribution homogeneous of degree $-n-\alpha$.

2. Algebras of s.i.o. with real analytic kernels on $\mathbf{R}^n$

We first describe the "compact" operators to be ignored; i.e., the operators whose symbol is $\equiv 0$.

Definition 2.1. Let k be a nonnegative integer and C a bounded operator on L^p. Then $C \in \mathscr{C}_k(\mathbf{R}^n)$ iff

(1) For $0 \le j \le k-1$, C and C^* are bounded from L^p_j to L^p_{j+1},
(2) C is a bounded operator on L^p_k and for every $\phi, \psi \in C^k_0(\mathbf{R}^n)$, the map $f \mapsto \phi C(\psi f)$ is a compact operator on L^p_k.

Our algebra of s.i.o. will consist of s.i.o. with kernels [symbols] in $\mathscr{C}^{k, \mathscr{O}}(\mathbf{R}^n)$ plus operators in $\mathscr{C}_k(\mathbf{R}^n)$.

Definition 2.2. Let $\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ denote the space of functions $F(x, z)$ which are homogeneous of degree 0 and in $\mathscr{C}^{k,\mathscr{O}}(\mathbf{R}^n)$.

An operator T on L^p is an *operator* in $\mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ iff

$$Tf(x) = \frac{1}{(2\pi)^n}\int e^{ix\xi}F(x, \xi)\hat{f}(\xi)\,d\xi + Cf(x), \tag{2–1}$$

where

(1) $F(x, \xi)$ is in $\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$,
(2) $C \in \mathscr{C}_k(\mathbf{R}^n)$.

The *symbol* of the operator T in (2–1) is

$$\sigma(T)(x, \xi) = F(x, \xi) = a_0(x) + \sum a_{lm}(x)Y_{lm}(\xi'). \tag{2–2}$$

The *kernel* of the operator T in (2–1) is

$$\begin{aligned} k_T(x, z) &= a_0(x)\delta(z) + k(x, z), \\ k(x, z) &= |z|^{-n}\sum a_{lm}(x)\gamma_m^{-1}Y_{lm}(z') \in \mathscr{C}^{k,\mathscr{O}}(\mathbf{R}^n). \end{aligned} \tag{2–3}$$

We note that if $T \in \mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$, then T is also a s.i.o. of type C_k^∞ as developed by Seeley [SCM]. Operators in $\mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ form a subalgebra of the singular integral operators of type C_k^∞; operators in $\mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ are bounded on L^p_j, $0 \le j \le k$. In particular the symbol and kernel are well defined and we have the usual symbolic calculus:

$$\begin{aligned} \sigma(T_1T_2)(x, \xi) &= \sigma(T_1)(x, \xi)\cdot\sigma(T_2)(x, \xi), \\ \sigma(T^*)(x, \xi) &= \bar{\sigma}(T)(x, \xi). \end{aligned} \tag{2–4}$$

Theorem 3 (Symbol Exact Sequence). *For each $k \ge 0$, the symbol map, σ : $\mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n) \to \mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$, is a $*$-algebra homomorphism, and the following sequence is exact*:

$$0 \to \mathscr{C}_k(\mathbf{R}^n) \to \mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n) \xrightarrow{\sigma} \mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n) \to 0.$$

3. C^1 COORDINATE CHANGES FOR $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$

We show that the symbol of an s.i.o. in $\mathrm{Op}\,\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ is invariantly defined under C^1 changes of coordinates. This step exploits the real analyticity of the kernels.

Let Ω and $\tilde{\Omega}$ be bounded open subsets of $\mathbf{R}^n$ and $\chi\colon \Omega \to \tilde{\Omega}$ a C^1 diffeomorphism. Assume that

$$0 < c < \|d\chi(x)\| < 1/c, \qquad x \in \Omega,$$

and we shall write $\chi(x) = \bar{x}$, etc.; $J(x) = |d\chi(x)|$ is the Jacobian of the transformation. For $f \in L^p(\Omega)$ let $(\chi_* f)(\bar{x}) = f(\chi^{-1}(\bar{x})) = \bar{f}(\bar{x})$ and for $\bar{g}(\bar{x}) \in L^p(\tilde{\Omega})$, let $(\chi^*\bar{g})(x) = \bar{g}(\chi(x)) = g(x)$.

Theorem 4. *Let* $T \in \operatorname{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$, *and suppose that the support of* $\sigma(T)$ *is in a compact subset* $\tilde{K}$ *of* $\tilde{\Omega}$ *and that* T *has kernel* $a_0(x)\delta(z) + k(x, z)$. *For* $f \in L^p(\Omega)$, *define*

(3–1)
$$\begin{aligned} T_\chi f(x) &= \chi^*(T(\chi_* f))(x) \\ &= a_0(\bar{x})f(x) + \lim_{\epsilon\to 0}\int_{|\bar{x}-\bar{y}|>\epsilon} k(\bar{x}, \bar{x}-\bar{y})\bar{f}(\bar{y})\,d\bar{y} \\ &= a_0(\chi(x))f(x) + \lim_{\epsilon\to 0}\int_{|\chi(x)-\chi(y)|>\epsilon} k(\chi(x), \chi(x)-\chi(y))f(y)J(y)\,dy. \end{aligned}$$

Then the operator T_χ *is in* $\operatorname{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ *and*

$$\sigma(T_\chi)(x, \xi) = \sigma(T)(\chi(x), [(d\chi(x))^{-1}]^t \cdot \xi). \tag{3–2}$$

The proof of Theorem 4 goes in two steps. Since multiplication by a_0 is no problem, we first consider the case where

$$T\bar{f}(\bar{x}) = \lim_{\epsilon\to 0}\int_{|\bar{x}-\bar{y}|>\epsilon} k(\bar{x}, \bar{x}-\bar{y})\bar{f}(\bar{y})\,d\bar{y}, \tag{3–3}$$

where $k(\bar{x}, -z) = -k(\bar{x}, z)$ is an odd kernel. We then handle the case of an even kernel using Riesz transforms and the symbolic calculus of Theorem 3.

Before treating odd kernels, we first handle some technical difficulties. If the support of $k(\bar{x}, \cdot)$ is $\tilde{K}$, let $K = \chi^{-1}(\tilde{K})$ and choose $\phi \in C_0^\infty(\Omega)$, $\phi \equiv 1$ on K. Replacing f by ϕf gives an operator in $\mathscr{C}_0(\mathbf{R}^n)$. Using a finite partition of unity on Ω, we may assume that $\|d\chi(x) - d\chi(x^0)\| \le \delta'\|d\chi(x^0)\|$, where δ' is a convenenient small constant. We shall therefore assume that $f(x)$ and $k(\chi(x), \cdot)$ have support near a point x^0.

Theorem 5 (The odd case). *Let* $k(\bar{x}, -z) = -k(\bar{x}, z)$ *be an odd kernel and*

$$\begin{aligned} T_\chi f(x) &= \chi^*(T(\chi_* f))(x) \\ &= \lim_{\epsilon\to 0}\int_{|\bar{x}-\bar{y}|>\epsilon} k(\bar{x}, \bar{x}-\bar{y})\bar{f}(\bar{y})\,d\bar{y} \\ &= \lim_{\epsilon\to 0}\int_{|\chi(x)-\chi(y)|>\epsilon} k(\chi(x), \chi(x)-\chi(y))f(y)J(y)\,dy, \\ \tilde{T}f(x) &= \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} k(\chi(x), d\chi(x)\cdot(x-y))f(y)J(y)dy. \end{aligned} \tag{3–4}$$

Then $T_\chi - \tilde{T} \in \mathscr{C}_0(\mathbf{R}^n)$ *and*

$$\sigma(T_\chi)(x, \xi) = \sigma(T)(\chi(x), [(d\chi(x))^{-1}]^t \cdot \xi). \tag{3–5}$$

Before proving Theorem 5, we note that the change of metric in (3–4) (from $|x-y| > \epsilon$ to $|\chi(x)-\chi(y)| > \epsilon$) has no effect on the limit in the case of odd kernels. See [SCM, Lemma 5] for the smooth case and the effect on even kernels. In the following lemma we introduce the metric $\rho(x, y) = |\chi(x) - \chi(y)|$, and let $\mathrm{M} f$ denote the Hardy-Littlewood maximal function of f.

Lemma 1. *Let $k(x, z)$ be an odd kernel. Consider the four operators*

$$(3\text{–}6)\quad \begin{aligned} K_\epsilon f(x) &= \int_{|x-y|>\epsilon} k(\chi(x), d\chi(x)\cdot(x-y)) f(y) J(y)\, dy, \\ K_{\epsilon,\rho} f(x) &= \int_{|\chi(x)-\chi(y)|>\epsilon} k(\chi(x), d\chi(x)\cdot(x-y)) f(y) J(y)\, dy, \\ \tilde{K}_\epsilon f(x) &= \int_{|x-y|>\epsilon} k(\chi(x), \chi(x)-\chi(y)) f(y) J(y)\, dy, \\ \tilde{K}_{\epsilon,\rho} f(x) &= \int_{|\chi(x)-\chi(y)|>\epsilon} k(\chi(x), \chi(x)-\chi(y)) f(y) J(y)\, dy. \end{aligned}$$

Then

(1) $\sup_{\epsilon>0} |K_\epsilon f(x) - K_{\epsilon,\rho} f(x)| \le c\,\mathrm{M} f(x)$.

(2) $\sup_{\epsilon>0} |\tilde{K}_\epsilon f(x) - \tilde{K}_{\epsilon,\rho} f(x)| \le c\,\mathrm{M} f(x)$.

(3) $\| \sup_{\epsilon>0} |K_\epsilon f(x)| \|_p \le c_p \|f\|_p$ *and* $\lim_{\epsilon\to 0} K_\epsilon f(x) = \tilde{T} f(x)$ *exists in* L^p *and a.e.*

(4) $\| \sup_{\epsilon>0} |\tilde{K}_{\epsilon,\rho} f(x)| \|_p \le c_p \|f\|_p$ *and* $\lim_{\epsilon\to 0} \tilde{K}_{\epsilon,\rho} f(x) = T_\chi f(x)$ *in* L^p *and a.e.*

(5) $\lim_{\epsilon\to 0}(K_\epsilon f(x) - K_{\epsilon,\rho} f(x)) = 0$ *in* L^p *and a.e.*

(6) $\lim_{\epsilon\to 0}(\tilde{K}_\epsilon f(x) - \tilde{K}_{\epsilon,\rho} f(x)) = 0$ *in* L^p *and a.e.*

Proof of Lemma 1. Inequalities (1) and (2) follow from the fact that the differences are expressed by integrals over a region contained in $c\epsilon \le |x-y| \le c^{-1}\epsilon$; estimating kernels by $C\epsilon^{-n}$ we obtain the bound

$$\int_{c\epsilon \le |x-y| \le c^{-1}\epsilon} C\epsilon^{-n} |f(y)|\, dy \le c\,\mathrm{M} f(x).$$

Inequalities (3) and (4) follow from bounds for the maximal singular integral operator for the s.i.o. with kernels $k(\bar{x}, z)$ and $k(\chi(x), d\chi(x)\cdot z)$ [CZ 1, CM].

To prove (5) and (6) it suffices to calculate the pointwise limit for $F(x) = f(x)J(x) \in C_0^1$. The first observation is that in calculating $K_{\epsilon,\rho} f(x)$ and $\tilde{K}_{\epsilon,\rho} f(x)$, we may use the uniform continuity of $d\chi$ to replace the region of integration $\{y : |\chi(x) - \chi(y)| > \epsilon\}$ by $\{y : |d\chi(x)\cdot(x-y)| > \epsilon\}$ with an error bounded by $c_\epsilon \sup |F(x)|$, where c_ϵ is small if ϵ is small. To prove (6), let

$$(3\text{–}7)\quad \begin{aligned} K_\epsilon^1 F(x) = &\int_{|x-y|>\epsilon} k(\chi(x), \chi(x)-\chi(y)) F(y)\, dy \\ &- \int_{|d\chi(x)\cdot(x-y)|>\epsilon} k(\chi(x), \chi(x)-\chi(y)) F(y)\, dy. \end{aligned}$$

Now use that $k(\chi(x), d\chi(x)\cdot z)$ is an odd kernel and that each integral in (3–7)

may be taken over a region contained in $c\epsilon \le |x-y| \le c^{-1}\epsilon$. We have
(3–8)

$$K_\epsilon^1 F(x) = \int_{\substack{|x-y|>\epsilon \\ c\epsilon \le |x-y| \le c^{-1}\epsilon}} [k(\chi(x), \chi(x)-\chi(y)) - k(\chi(x), d\chi(x)\cdot(x-y))](F(y)-F(x))\,dy$$

$$- \int_{\substack{|d\chi(x)\cdot(x-y)|>\epsilon \\ c\epsilon \le |x-y| \le c^{-1}\epsilon}} [k(\chi(x), \chi(x)-\chi(y)) - k(\chi(x), d\chi(x)\cdot(x-y))](F(y)-F(x))\,dy$$

$$+ F(x) I_\epsilon(x).$$

Each integral in (3–8) is absolutely convergent for $F(x) \in C_0^1$ and the limit of the difference of the integrals is 0. To estimate $I_\epsilon(x)$, we use the continuity of $d\chi(x)$ to obtain that

$$|k(\chi(x), \chi(x)-\chi(y)) - k(\chi(x), d\chi(x)\cdot(x-y))| \le c_\epsilon |d\chi(x)\cdot(x-y)||x-y|^{-n-1} \quad \text{for} \quad c\epsilon \le |x-y| \le c^{-1}\epsilon.$$

It follows that $|I_\epsilon(x)| \le c_\epsilon \int_{c\epsilon \le |x-y| \le c^{-1}\epsilon} |x-y|^{-n}\,dy \to 0$ as $\epsilon \to 0$.

The proof of (5) is similar. □

Proof of Theorem 5. Let

$$\tilde{K} f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} k(\chi(x), \chi(x)-\chi(y)) f(y) J(y)\,dy$$

(3–9)
$$= \lim_{\epsilon\to 0} \tilde{K}_\epsilon f(x) = T_\chi f(x),$$

$$K f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} k(\chi(x), d\chi(x)\cdot(x-y)) f(y) J(y)\,dy.$$

We will show that $\tilde{K} - K \in \mathscr{C}_0(\mathbf{R}^n)$. Let $\{\chi^j(x)\}$ be a sequence of C^∞ coordinate changes converging to $\chi(x)$ in C^1. If

(3–10)
$$\tilde{K}^j f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} k(\chi(x), \chi^j(x)-\chi^j(y)) f(y) J(y)\,dy,$$

$$K^j f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} k(\chi(x), d\chi^j(x)\cdot(x-y)) f(y) J(y)\,dy,$$

and $C^j = \tilde{K}^j - K^j$, then $C^j \in \mathscr{C}_0(\mathbf{R}^n)$ [SCM, Lemma 6, p. 675]. To show that $\lim C^j = (\tilde{K} - K)$ in operator norm, we shall apply the Multilinear Commutator Theorem of [CMM, Theorem 3] to show that

(3–11)
$$\left\| \sup_{\epsilon>0} |\tilde{K}_\epsilon^j f(x) - \tilde{K}_\epsilon f(x)| \right\| \le c_p \|d\chi^j - d\chi\|_\infty \|f\|_p,$$

$$\left\| \sup_{\epsilon>0} |K_\epsilon^j f(x) - K_\epsilon f(x)| \right\| \le c_p \|d\chi^j - d\chi\|_\infty \|f\|_p.$$

Let $F(x) = f(x)J(x)$; using a partition of unity assume that the support of F is near x^0. We choose $\delta > 0$ so small that $k(\chi(x), Z)$ is an analytic function of Z for $Z \in \mathbf{C}^n$, $|Z_i - (d\chi(x^0)\cdot z)_i| \leq 100\delta|d\chi(x^0)\cdot z|$, $z \in \mathbf{R}^n\setminus\{0\}$.

We apply the method of rotation [CZ 2] to

(3–12)
$$\tilde{T}_\epsilon^j F(x) = \int_{|x-y|>\epsilon} [k(\chi(x), \chi^j(x) - \chi^j(y)) - k(\chi(x), \chi(x) - \chi(y))]F(y)\,dy.$$

We write

(3–13)
$$\begin{aligned}\tilde{T}_\epsilon^j F(x) = \frac{1}{2}\Bigg\{ &\int_{|z|>\epsilon} [k(\chi(x), \chi^j(x) - \chi^j(x-z)) \\ &\qquad - k(\chi(x), \chi(x) - \chi(x-z))]F(x-z)\,dz \\ &+ \int_{|z|>\epsilon} [k(\chi(x), \chi^j(x) - \chi^j(x+z)) \\ &\qquad - k(\chi(x), \chi(x) - \chi(x+z))]F(x+z)\,dz\Bigg\}.\end{aligned}$$

Using polar coordinates, $z = \rho\sigma$, $dz = \rho^{n-1}\,d\rho\;\;d\sigma$, and the homogeneity and oddness of k, we have

$$\tilde{T}_\epsilon^j F(x) = \frac{1}{2}\int_\Sigma T_{\sigma,\epsilon}^j F(x)\,d\sigma,$$

$$\begin{aligned}T_{\sigma,\epsilon}^j F(x) = \int_\epsilon^\infty \Bigg\{&\Bigg[k\left(\chi(x), \frac{\chi^j(x) - \chi^j(x-\rho\sigma)}{\rho}\right) \\ &-k\left(\chi(x), \frac{\chi(x) - \chi(x-\rho\sigma)}{\rho}\right)\Bigg] F(x-\rho\sigma) \\ &-\Bigg[k\left(\chi(x), \frac{\chi^j(x+\rho\sigma) - \chi^j(x)}{\rho}\right) \\ &-k\left(\chi(x), \frac{\chi(x+\rho\sigma) - \chi(x)}{\rho}\right)\Bigg] F(x+\rho\sigma)\Bigg\}\,\frac{d\rho}{\rho}.\end{aligned} \qquad (3\text{–}14)$$

Now fix σ and write $x = t\sigma + w$ where $w \perp \sigma$ so that

$$\int_{\mathbf{R}^n} |G(x)|^p\,dx = \int_{\mathbf{R}^{n-1}}\int_{-\infty}^{\infty} |G(t\sigma + w)|^p\,dt\,dw.$$

The first estimate of (3–11) will be shown if

$$\int \sup_{\epsilon>0} |T_{\sigma,\epsilon}^j F(t\sigma + w)|^p\,dt \leq c_p^p \|d\chi^j - d\chi\|_\infty^p \int |F(t\sigma + w)|^p\,dt. \qquad (3\text{–}15)$$

For fixed σ let $Z^0 = d\chi(x^0)\cdot\sigma$ and assume that

$$\|d\chi(x) - d\chi(x^0)\| \leq \delta\|d\chi(x^0)\|.$$

The kernel $k(\chi(x), Z)$ in (3–14) is evaluated at points $Z = (Z_1, \dots, Z_n)$ where $|Z_i - Z_i^0| \leq \delta$. Thus we use the contour integral representation for $k(\chi(x), Z)$ on the contours $|\zeta_i - Z_i^0| = 50\delta$:

(3–16)
$$k(\chi(x), Z) = \int_{|\zeta_i - Z_i^0| = 50\delta} \cdots \int k(\chi(x), \zeta) \frac{1}{P(\zeta - Z)} \frac{d\zeta_1 \cdots d\zeta_n}{(2\pi i)^n},$$
$$P(\zeta - Z) = \prod_{i=1}^{n} (\zeta_i - Z_i).$$

We introduce the notation

$$\Delta_{\sigma, w}\chi(t, s) = \frac{\chi(t\sigma + w) - \chi(s\sigma + w)}{t - s}$$

and write the integral in (3–16) as

(3–17)
$$k(\chi(x), Z) = \oint k(\chi(x), \zeta) \frac{1}{P(\zeta - Z)}\, d\mu(\zeta).$$

Thus

(3–18)
$$T^j_{\sigma,\epsilon} F(t\sigma + w) = \oint k(\chi(t\sigma + w), \zeta) T^j_{\sigma,\epsilon,\zeta} F(t\sigma + w)\, d\mu(\zeta),$$
$$\begin{aligned} T^j_{\sigma,\epsilon,\zeta} F(t\sigma + w) = \int_\epsilon^\infty \Big\{ & \left[\frac{1}{P(\zeta - \Delta_{\sigma,w}\chi^j(t, t-\rho))} - \frac{1}{P(\zeta - \Delta_{\sigma,w}\chi(t, t-\rho))} \right] \\ & \times F((t-\rho)\sigma + w) \\ & - \left[\frac{1}{P(\zeta - \Delta_{\sigma,w}\chi^j(t+\rho, t))} - \frac{1}{P(\zeta - \Delta_{\sigma,w}\chi(t+\rho, t))} \right] \\ & F((t+\rho)\sigma + w) \Big\} \frac{d\rho}{\rho}. \end{aligned}$$

Of course

(3–19)
$$\frac{1}{P(\zeta - Z^j)} - \frac{1}{P(\zeta - Z)} = \frac{-\sum_{1 \le |\alpha| \le n} (\alpha!)^{-1} (D^\alpha P)(\zeta - Z)(Z - Z^j)^\alpha}{P(\zeta - Z^j) P(\zeta - Z)}$$

and

(3–20)
$$\frac{1}{P(\zeta - Z)} = \frac{1}{P(\zeta - Z^0)} \frac{1}{\prod \left(1 - \frac{Z_i - Z_i^0}{\zeta_i - Z_i^0}\right)} = \frac{1}{P(\zeta - Z^0)} \sum_{\alpha \ge 0} \frac{(Z - Z^0)^\alpha}{(\zeta - Z^0)^\alpha}.$$

Using (3–19) to simplify the kernels in (3–18) and then expanding $1/P(\zeta - \cdot)$ in a power series using (3–20), we apply the Multilinear Commutator Theorem of [CMM] to obtain that

(3–21)
$$\int \sup_{\epsilon > 0} |T^j_{\sigma,\epsilon,\zeta} F(t\sigma + w)|^p\, dt \le c_p^p A_\zeta^p \int |F(t\sigma + w)|^p\, dt.$$

with

(3–22)
$$A_\zeta = c \sum_{1 \le |\alpha| \le n} c_\alpha \|d\chi^j - d\chi\|_\infty^{|\alpha|} \left(\sum_{m=0}^{\infty} (1+m)^4 \left(\frac{c\delta'}{49\delta} \right)^m \right)^2.$$

Now integrate the estimates for $T^j_{\sigma,\epsilon,\zeta}$ with respect to $d\mu(\zeta)$ in (3–18) to obtain (3–15) and the first estimate in (3–11).

The proof of the second inequality in (3–11) proceeds in the same (but simpler) manner.

To calculate the symbol of the operator $\tilde{T}$ of (3–4), use that the symbol is the same as for the operator with kernel $k(\chi(x), d\chi(x)\cdot z)J(x)$. Then use a variant of Lemma 1 to show that

$$\begin{aligned}\sigma(\tilde{T})(x, \xi) &= \lim_{\epsilon\to 0, N\to\infty} \int_{\epsilon<|z|<N} e^{-iz\xi} k(\chi(x), d\chi(x)\cdot z)J(x)\, dz \\ &= \lim_{\epsilon\to 0, N\to\infty} \int_{\epsilon<|\chi(x)\cdot z|<N} e^{-iz\xi} k(\chi(x), d\chi(x)\cdot z)J(x)\, dz \\ &= \sigma(T)(\chi(x), [(d\chi(x))^{-1}]^t\cdot\xi).\end{aligned} \tag{3–23}$$

This proves Theorem 5. □

Proof of Theorem 4. It remains to treat the even kernel case. Let

$$T\bar{f}(\bar{x}) = \lim_{\epsilon\to 0}\int_{|\bar{x}-\bar{y}|>\epsilon} k(\bar{x}, \bar{x}-\bar{y})\bar{f}(\bar{y})\, d\bar{y}, \tag{3–24}$$

where $k(\bar{x}, -z) = k(\bar{x}, z)$ is an even kernel and $k(\bar{x}, \cdot)$ has compact support in $\tilde{\Omega}$. As in [CZ 2] we use the Riesz transforms. Let $R_j(z) = c_n z_j/|z|^{n+1}$ be the kernel of the jth Riesz transform so that $\sigma(R_j)(\bar{x}, \xi) = i\xi_j/|\xi|$. Then

$$\begin{aligned}\sigma(T)(\bar{x}, \xi) &= \sum_{j=1}^{n}\left(-\sigma(T)(\bar{x}, \xi)\cdot i\frac{\xi_j}{|\xi|}\right)\left(i\frac{\xi_j}{|\xi|}\right) \\ &= \sum_{j=1}^{n}\sigma(T_j)(\bar{x}, \xi)\sigma(R_j)(\bar{x}, \xi).\end{aligned} \tag{3–25}$$

If $k_j(\bar{x}, z)$ is the kernel of the operator T_j, then

$$T\bar{f}(\bar{x}) = \sum_{j=1}^{n}\lim_{\epsilon\to 0}\int_{|\bar{x}-\bar{y}|>\epsilon} k_j(\bar{x}, \bar{x}-\bar{y})(R_j\bar{f})(\bar{y})\, d\bar{y}. \tag{3–26}$$

Let $\bar{\phi}\in C_0^\infty(\tilde{\Omega})$, $\bar{\phi}\equiv 1$ on $\operatorname{supp} k(\bar{x}, \cdot)$ and let $R'_j = \bar{\phi}R_j\bar{\phi}$. Then

$$T = \sum_{j=1}^{n} T_j R'_j + \bar{\phi}C', \tag{3–27}$$

with $C'\in\mathscr{C}_0(\mathbf{R}^n)$, and

$$\begin{aligned}T_\chi &= \chi^*\left(\sum_{j=1}^{n} T_j R'_j\right)\chi_* + \chi^*\bar{\phi}C'\chi_* \\ &= \sum_{j=1}^{n}(\chi^* T_j\chi_*)(\chi^* R'_j\chi_*) + \chi^*\bar{\phi}C'\chi_*.\end{aligned} \tag{3–28}$$

By Theorem 5 and Theorem 3, $(T_j)_\chi$, $(R'_j)_\chi$, and T_χ are in $\mathrm{Op}\,\mathscr{C}_{(0)}^{0,\mathscr{O}}(\mathbf{R}^n)$ and

(3–29)

$$\begin{aligned}\sigma(T_\chi)(x, \xi) &= \sum_{j=1}^{n}(\sigma(\chi^* T_j\chi_*)(x, \xi))(\sigma(\chi^* R'_j\chi_*)(x, \xi)) \\ &= \sum_{j=1}^{n}\sigma(T_j)(\chi(x), [(d\chi(x))^{-1}]^t\cdot\xi)\,\sigma(R'_j)(\chi(x), [(d\chi(x))^{-1}]^t\cdot\xi) \\ &= \sigma(T)(\chi(x), [(d\chi(x))^{-1}]^t\cdot\xi).\end{aligned}$$

This concludes the proof of Theorem 4. □

Remark. E. B. Fabes pointed out that Theorem 5 (and hence Theorem 4) are valid for operators with kernels in $\mathscr{C}^{0,\infty}_{(-n)}$. For the proof write the odd kernel

$$k(\bar{x}, z') = \sum_{1 \le m \le N} a_{lm}(\bar{x}) Y_{lm}(z') + \sum_{N+1 \le m} a_{lm}(\bar{x}) Y_{lm}(z')$$

and the corresponding operator as $T = T_{1,N} + T_{N+1,\infty}$. Then $(T_{1,N})_\chi - \tilde{T}_{1,N}$ is a compact operator on $L^p(\Omega)$ and for N large $(T_{N+1,\infty})_\chi$ and $\tilde{T}_{N+1,\infty}$ have small norm on $L^p(\Omega)$. Hence $T_\chi - \tilde{T}$ is a compact operator on $L^p(\Omega)$. As a consequence the symbol of an operator in $\mathrm{Op}\,\mathscr{C}^{0,\infty}_{(0)}(\mathbf{R}^n)$ is invariantly defined under C^1 coordinate changes.

4. Singular integral operators on a C^1 manifold

Let $\mathscr{M}$ be a compact oriented manifold of dimension n and of class C^r, $r \ge 1$.

Definition 4.1. An operator C on $L^p(\mathscr{M})$ is in $\mathscr{C}_k(\mathscr{M})$, $0 \le k \le r-1$, iff

(1) For $0 \le j \le k-1$, C and C^* map $L^p_j(\mathscr{M})$ into $L^p_{j+1}(\mathscr{M})$.
(2) C is a compact operator on $L^p_k(\mathscr{M})$.

An operator T on $L^p(\mathscr{M})$ is in $\mathrm{Op}\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathscr{M})$ iff

(1) For each $\phi, \psi \in C^r_0(\mathscr{M})$ with disjoint support, $\phi T \psi \in \mathscr{C}_k(\mathscr{M})$,
(2) For each $\phi, \psi \in C^r_0(\mathscr{M})$ with support in a coordinate patch U with coordinates $x : U \mapsto \tilde{U} \subset \mathbf{R}^n$,

$$x^*(\phi T \psi) x_* \in \mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n). \tag{4–1}$$

If $T \in \mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$, the symbol of T is the function on the cotangent bundle of $\mathscr{M}$ defined by

$$\sigma(T)(p, \textstyle\sum \xi_i\, dx_i) = \sigma(x^*(\phi T \phi) x_*)(x(p), \xi) \tag{4–2}$$

where x is a coordinate function on a neighborhood U of p and $\phi \in C^r_0(U)$, $\phi \equiv 1$ near p.

It follows from Theorem 4 that the symbol of an operator $T \in \mathrm{Op}\,\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathbf{R}^n)$ is well defined. Let $\mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathscr{M})$ denote the space of functions on the cotangent bundle of $\mathscr{M}$ which are homogeneous of degree 0 and real analytic in the cotangent variables, and of class C^k with respect to the variables in $\mathscr{M}$. Then $\sigma(T) \in \mathscr{C}^{k,\mathscr{O}}_{(0)}(\mathscr{M})$. Following, e.g., [P], we then have the symbolic calculus, at least at the principal symbol level, for $\mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M})$. For completeness we record two results—the Symbol Exact Sequence and the characterization of elliptic (Fredholm) operators in $\mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M})$.

Theorem 6 (Symbol Exact Sequence). *Let $\mathscr{M}$ be a compact oriented manifold of class C^1. The symbol map, $\sigma : \mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M}) \to \mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M})$, is a $*$-algebra* *homomorphism, and the following sequence is exact:*

$$0 \to \mathscr{C}_0(\mathscr{M}) \to \mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M}) \xrightarrow{\sigma} \mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M}) \to 0.$$

Theorem 7. *Let* $T = (T_{ij})$ *be an* $N \times N$ *system of operators in* $\mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M})$. *The following are equivalent:*

(1) $\big(\sigma(T_{ij})(p, \sum \xi_i\, dx_i)\big)$ *is a nonsingular matrix for each* $p \in \mathscr{M}$, $\sum \xi_i\, dx_i \neq 0$.

(2) *There is a bounded operator* S *on* $L^p(\mathscr{M})$ *such that* $ST = I + C$, $C \in \mathscr{C}_0(\mathscr{M})$.

(3) *There is an a priori estimate*

$$\|f\|_p \le c\,\|Tf\|_p + \|Cf\|_p, \tag{4–3}$$

with $C \in \mathscr{C}_0(\mathscr{M})$.

Proof of Theorem 7. If (1) holds, let $\sigma(S) = [\sigma(T)]^{-1}$ and apply Theorem 6 to obtain (2). If (2) holds, write $f = STf - Cf$ to obtain (3).

Suppose that inequality (4–3) holds. Fix $p \in \mathscr{M}$, U a coordinate neighborhood of p with coordinates $x : U \mapsto \tilde{U} \subset \mathbf{R}^n$ and fix $\phi \in C^1_0(U)$, $\phi \equiv 1$ near p. Let $\tilde{T} = x^*(\phi T \phi)x_* \in \mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathbf{R}^n)$. Then for $g \in C^\infty_0$ with support near $x(p)$,

$$T(x^* g) = x^*(\tilde{T} g) + C(x^* g) \tag{4–4}$$

so that

$$\|g\|_p \le c'\|\tilde{T} g\|_p + c'\|C' g\|_p, \tag{4–5}$$

with C' compact on $L^p(\mathbf{R}^n)$. We are now reduced to the Euclidean case and we fix $\xi^0 \neq 0$ and apply the inequality (4–5) to $g_k(x) = e^{ikx\xi^0} g(x)$. We have that $\|g_k\|_p = \|g\|_p$, $\|C' g_k\|_p \to 0$ as $k \to \infty$, and

$$\begin{aligned}\lim_{k\to\infty} e^{-ikx\xi^0}\tilde{T} g_k(x) &= \lim_{k\to\infty} \frac{1}{(2\pi)^n}\int e^{ix\xi}\sigma(\tilde{T})(x, \xi + k\xi^0)\hat{g}(\xi)\,d\xi \\ &= \frac{1}{(2\pi)^n}\int e^{ix\xi}\sigma(\tilde{T})(x, \xi^0)\hat{g}(\xi)\,d\xi \\ &= \sigma(\tilde{T})(x, \xi^0) g(x).\end{aligned} \tag{4–6}$$

The limits in (4–6) are taken in $L^p(\mathbf{R}^n)$. Hence

$$\|g\|_p \le c'\|\sigma(\tilde{T})(\cdot, \xi^0) g\|_p \tag{4–7}$$

for all g with support near $x(p)$. This implies that $\sigma(\tilde{T})(x(p), \xi^0)$ is a nonsingular matrix. □

In particular a system of operators, $T = (T_{ij})$, in $\mathrm{Op}\mathscr{C}^{0,\mathscr{O}}_{(0)}(\mathscr{M})$ is Fredholm on $L^p(\mathscr{M})$ iff $\big(\sigma(T_{ij})(p, \sum \xi_i\, dx_i)\big)$ is a nonsingular matrix for each $p \in \mathscr{M}$, $\sum \xi_i\, dx_i \neq 0$; such an operator [system] is called elliptic.

References

[B 1] L. Boutet de Monvel, *Opérateurs pseudo-differentiels analytique et problémes aux limites elliptiques*, Ann. Inst. Fourier (Grenoble) **19** (1969), 169–269.

[B 2] ——, *Opérateurs pseudo-differentiels analytique et opérateurs d'ordre infini*, Ann. Inst. Fourier (Grenoble) **22** (1972), 229–269.

[BK] L. Boutet de Monvel and P. Krée, *Pseudodifferential operators and Gevrey classes*, Ann. Inst. Fourier (Grenoble) **17** (1967), 295–323.

[CZ 1] A. P. Calderón and A. Zygmund, *Singular integral operators and differential equations*, Amer. J. Math. **79** (1957), 901–921.

[CZ 2] ——, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[CG] J. Cohen and J. Gosselin, *The Dirichlet problem for the biharmonic equation in a C^1 domain in the plane*, Indiana Univ. Math. J. **34** (1983), 635–685.

[CMM] R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Ann. of Math. **116** (1982), 361–387.

[CM] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque **57** (1978).

[Cos] M. Costabel, *Singular integral operators on curves with corners*, Integral Equations Operator Theory **3** (1980), 323–349.

[FJR] E. B. Fabes, M. Jodeit, Jr., and N. M. Rivière, *Potential techniques for boundary value problems on C^1 domains*, Acta Math. **141** (1977), 263–291.

[L] J. E. Lewis, *A symbolic calculus for layer potentials on C^1 curves and C^1 curvilinear polygons*, Proc. Amer. Math. Soc. **112** (1991), 419–427.

[LP] J. E. Lewis and C. Parenti, *Pseudodifferential operators of Mellin type*, Comm. Partial Differential Equations **8** (1983), 477–544.

[N] Umberto Neri, *Singular integrals*, Lecture Notes in Math., vol. 200, Springer-Verlag, Berlin and New York, 1971.

[P] R. S. Palais, *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Stud., no. 57, Princeton Univ. Press, 1965.

[SCM] R. T. Seeley, *Singular integrals on compact manifolds*, Amer. J. Math. **81** (1959), 658–690.

[SS 1] R. Selvaggi and I. Sisto, *Regolarità di certe trasformazioni integrali relative ad aperti di classe C^1*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **45** (1978), 393–410.

[SS 2] ——, *Problemi al contorno per sistemi ellitici simmetrici del primo ordine a coefficienti costanti in aperti di classe C^1*, Note Mat. **1** (1981), 155–185.

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60607-7045

E-mail address: u12585@uicvm.bitnet

Dipartimento di Matematica, Università di Lecce, Via Arnesana, 73100 Lecce, Italy

Dipartimento di Matematica, Università di Bari, Via Re David, 70125 Bari, Italy

E-mail address: matem@vm.csata.it